WORKED EXAMPLE 14

Variables V and t are connected by the equation $V = 5t^2 - 8t + 3$.

Find the rate of change of V with respect to t when $t = 4$.

Answers

$$V = 5t^2 - 8t + 3$$

$$\frac{dV}{dt} = 10t - 8$$

When $t = 4$, $\frac{dV}{dt} = 10(4) - 8 = 32$

Connected rates of change

When two variables x and y both vary with a third variable t, the three variables can be connected using the chain rule:

$$\frac{dy}{dt} = \frac{dy}{dx} \times \frac{dx}{dt}$$

You may also need to use the rule that:

$$\frac{dx}{dy} = \frac{1}{\frac{dy}{dx}}$$

WORKED EXAMPLE 15

Variables x and y are connected by the equation $y = x^3 - 5x^2 + 15$.
Given that x increases at a rate of 0.1 units per second, find the rate of change of y when $x = 4$.

Answers

$y = x^3 - 5x^2 + 15$ and $\frac{dx}{dt} = 0.1$

$\frac{dy}{dx} = 3x^2 - 10x$

When $x = 4$, $\frac{dy}{dx} = 3(4)^2 - 10(4)$

$= 8$

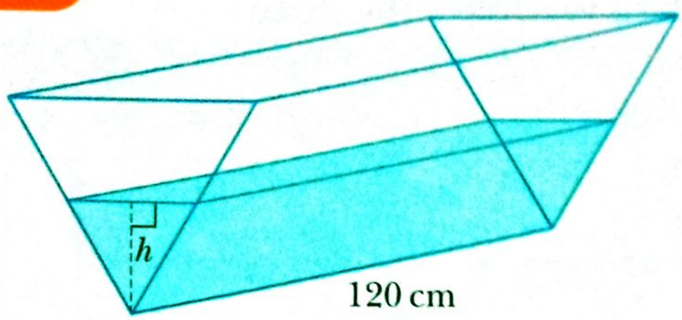
Using the chain rule, $\frac{dy}{dt} = \frac{dy}{dx} \times \frac{dx}{dt}$

$= 8 \times 0.1$

$= 0.8$

Rate of change of y is 0.8 units per second.

WORKED EXAMPLE 16



The diagram shows a water container in the shape of a triangular prism of length 120 cm.

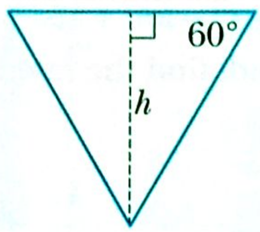
The vertical cross-section is an equilateral triangle.

Water is poured into the container at a rate of $24\,\text{cm}^3\,\text{s}^{-1}$.

a Show that the volume of water in the container, $V\,\text{cm}^3$, is given by $V = 40\sqrt{3}\,h^2$, where h cm is the height of the water in the container.

b Find the rate of change of h when $h = 12$.

Answers



a Length of side of triangle $= \dfrac{h}{\sin 60°}$

$$= \frac{2\sqrt{3}h}{3}$$

Area of triangle $= \dfrac{1}{2} \times \dfrac{2\sqrt{3}h}{3} \times h$

$$= \frac{\sqrt{3}h^2}{3}$$

V = area of triangle × 120

$$= \frac{\sqrt{3}h^2}{3} \times 120$$

$$= 40\sqrt{3}\,h^2$$

b $\dfrac{dV}{dh} = 80\sqrt{3}\,h$ and $\dfrac{dV}{dt} = 24$

When $h = 12$, $\dfrac{dV}{dh} = 80\sqrt{3}\,(12)$

$$= 960\sqrt{3}$$

Using the chain rule, $\dfrac{dh}{dt} = \dfrac{dh}{dV} \times \dfrac{dV}{dt}$

$$= \frac{1}{960\sqrt{3}} \times 24$$

$$= \frac{\sqrt{3}}{120}$$

Rate of change of h is $\dfrac{\sqrt{3}}{120}$ cm per second.

Exercise 12.7

1 Variables x and y are connected by the equation $y = x^2 - 5x$.
Given that x increases at a rate of 0.05 units per second, find the rate of change of y when $x = 4$.

2 Variables x and y are connected by the equation $y = x + \sqrt{x - 5}$.
Given that x increases at a rate of 0.1 units per second, find the rate of change of y when $x = 9$.

3 Variables x and y are connected by the equation $y = (x - 3)(x + 5)^3$.
Given that x increases at a rate of 0.2 units per second, find the rate of change of y when $x = -4$.

4 Variables x and y are connected by the equation $y = \frac{5}{2x - 1}$.
Given that y increases at a rate of 0.1 units per second, find the rate of change of x when $x = -2$.

5 Variables x and y are connected by the equation $y = \frac{2x}{x^2 + 3}$.
Given that x increases at a rate of 2 units per second, find the rate of increase of y when $x = 1$.

6 Variables x and y are connected by the equation $y = \frac{2x - 5}{x - 1}$.
Given that x increases at a rate of 0.02 units per second, find the rate of change of y when $y = 1$.

7 Variables x and y are connected by the equation $\frac{1}{y} = \frac{1}{8} - \frac{2}{x}$.
Given that x increases at a rate of 0.01 units per second, find the rate of change of y when $x = 8$.

8 A square has sides of length x cm and area $A\,\text{cm}^2$.
The area is increasing at a constant rate of $0.2\,\text{cm}^2\,\text{s}^{-1}$.
Find the rate of increase of x when $A = 16$.

9 A cube has sides of length x cm and volume $V\,\text{cm}^3$.
The volume is increasing at a rate of $2\,\text{cm}^3\,\text{s}^{-1}$.
Find the rate of increase of x when $V = 512$.

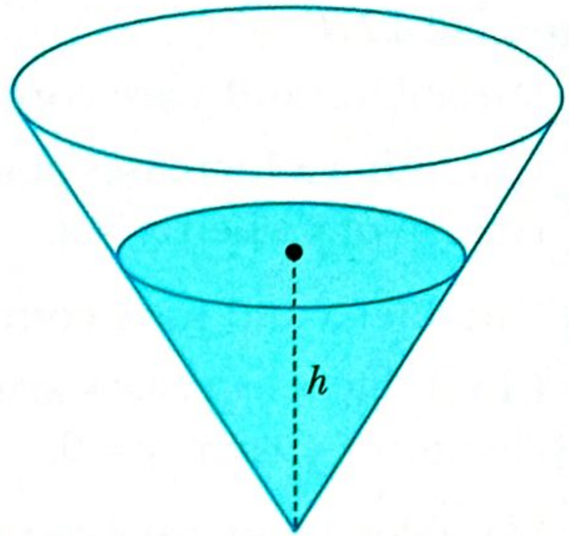
10 A sphere has radius r cm and volume $V\,\text{cm}^3$.
The radius is increasing at a rate of $\frac{1}{\pi}\,\text{cm}\,\text{s}^{-1}$.
Find the rate of increase of the volume when $V = 972\pi$.

11 A solid metal cuboid has dimensions x cm by x cm by $5x$ cm.
The cuboid is heated and the volume increases at a rate of $0.5\,\text{cm}^3\,\text{s}^{-1}$.
Find the rate of increase of x when $x = 4$.

12 A cone has base radius r cm and a fixed height 18 cm.
The radius of the base is increasing at a rate of $0.1\,\text{cm}\,\text{s}^{-1}$.
Find the rate of change of the volume when $r = 10$.

13 Water is poured into the conical container at a rate of $5\,\text{cm}^3\,\text{s}^{-1}$.

After t seconds, the volume of water in the container, $V\,\text{cm}^3$, is given by $V = \dfrac{1}{12}\pi h^3$, where h cm is the height of the water in the container.

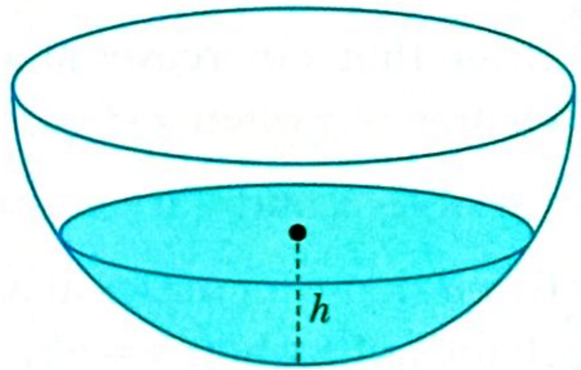


a Find the rate of change of h when $h = 5$.

b Find the rate of change of h when $h = 10$.

14 Water is poured into the hemispherical bowl at a rate of $4\pi\,\text{cm}^3\,\text{s}^{-1}$.

After t seconds, the volume of water in the bowl, $V\,\text{cm}^3$, is given by $V = 8\pi h^2 - \dfrac{1}{3}\pi h^3$, where h cm is the height of the water in the bowl.



a Find the rate of change of h when $h = 2$.

b Find the rate of change of h when $h = 4$.

g $-\frac{15}{(3x-1)^2}$ h $-\frac{x^2+8x+2}{\left(x^2-2\right)^2}$

2 -4

3 $(0, 0)$, $(1, 1)$

4 $\frac{25}{9}$

5 a $\frac{1-2x}{2\sqrt{x}\,(2x+1)^2}$

b $\frac{1-x}{(1-2x)^{\frac{3}{2}}}$

c $\frac{x\left(x^2+4\right)}{\left(x^2+2\right)^{\frac{3}{2}}}$

d $\frac{-5(x-3)}{2\sqrt{x}\,(x+3)^2}$

6 4

7 $(3, -2)$

8 a $(-2, -2.4)$, $(0.4, 0)$, $(2, 1.6)$

b $-\frac{23}{25}, \frac{125}{29}, -\frac{7}{25}$

Exercise 12.5

1 a $y = 4x - 6$

b $y = -x - 2$

c $y = 16x - 10$

d $y = -\frac{1}{2}x + 3$

e $y = -3x - 3$

f $y = \frac{1}{4}x + 2\frac{1}{4}$

2 a $y = -\frac{1}{3}x - 4\frac{1}{3}$

b $y = -\frac{1}{8}x + 5\frac{1}{4}$

c $y = \frac{1}{4}x - 3\frac{3}{4}$

d $y = 2x + 7.5$

e $y = -0.1x - 3.8$

f $y = 4x - 22$

3 $y = 8x - 6$, $y = -\frac{1}{8}x + \frac{17}{8}$

4 $(0, 5.2)$

5 $y = \frac{9}{16}x - \frac{1}{2}$, $y = -\frac{16}{9}x - \frac{1}{2}$

6 $(2, -3)$

7 $y = 2x - 20$

8 a $y = x + 8$

b $(1, 6)$

c $y = -\frac{1}{2}x + \frac{13}{2}$

9 $(1, 5.25)$

10 a $(7, 4)$

b 12 units^2

11 b $y = -0.4x - 0.6$

12 216 units^2

13 22.5 units^2

Exercise 12.6

1 0.21 2 0.68

3 $-0.8p$ 4 $2p$

5 $25p$ 6 $\frac{11}{3}p$

7 $\frac{\pi}{20}$

8 a $y = \frac{180}{x^2}$

c $-254p$, decrease

Exercise 12.7

1 0.15 units per second

2 0.125 units per second

3 -4 units per second

4 -0.25 units per second

5 0.5 units per second

6 $\frac{1}{150}$ units per second

7 -0.08 units per second

8 0.025 cm s^{-1}

9 $\frac{1}{96}$ cm s^{-1}

10 324 cm^3 s^{-1}

11 $\frac{1}{480}$ cm s^{-1}

12 12π cm^3 s^{-1}

13 a $\frac{4}{5\pi}$ cm s^{-1} b $\frac{1}{5\pi}$ cm s^{-1}

14 a $\frac{1}{7}$ cm s^{-1} b $\frac{1}{12}$ cm s^{-1}

Exercise 12.8

1 a 10 b $12x + 6$

c $-\frac{18}{x^4}$ d $320(4x+1)^3$

e $-\frac{1}{(2x+1)^{\frac{3}{2}}}$ f $\frac{3}{(x+3)^{\frac{5}{2}}}$

2 a $12(x-4)(x-2)$

b $\frac{8x-6}{x^4}$

c $\frac{8}{(x-3)^3}$

d $\frac{2\left(x^3+6x^2+3x+2\right)}{\left(x^2-1\right)^3}$

e $y = \frac{50}{(x-5)^3}$

f $y = \frac{102}{(3x-1)^3}$

3 a -3 b -9 c -8

4 b -18, 18

5

x	0	1	2	3	4	5
$\frac{dy}{dx}$	+	0	–	–	0	+
$\frac{d^2y}{dx^2}$	–	–	–	+	+	+

6 $x > 2$

Exercise 12.9

1 a $(6, -28)$ minimum

b $(-2, 9)$ maximum

c $(-2, 18)$ maximum, $(2, -14)$ minimum

d $\left(-2\frac{2}{3}, 14\frac{22}{27}\right)$ maximum, $(2, -36)$ minimum

e $(-3, -18)$ minimum, $\left(\frac{1}{3}, \frac{14}{27}\right)$ maximum